SOLUTION OF INVERSE PROBLEMS FOR THE QUASILINEAR HEAT CONDUCTION EQUATION IN THE SELF-SIMILAR MODE FOR THE MULTIDIMENSIONAL CASE

T. B. Gardashov UDC 536.24

Explicit solutions are found for inverse problems for the quasilinear heat conduction equation in the case of self-similarity of the process for the multidimensional case. The unknown thermophysical characteristics depend on the temperature distribution.

Mathematical modeling of nonstationary heat transfer processes is closely connected with the solution of inverse problems concerned with the determination of nonlinear thermophysical characteristics of the mathematical mode; these characteristics are coefficients of a quasilinear heat conduction equation. References [1-4] are concerned with differing approaches to inverse problems of the determination of nonlinear thermophysical characteristics of mathematical models. In [5-9] self-similar solutions are employed to determine nonlinear thermophysical characteristics for a mathematical model, these characteristics being the coefficients of a one-dimensional quasilinear heat conduction equation, and also systems of quasilinear heat conduction equations. In the present paper, in contrast to the papers mentioned, we consider a multidimensional quasilinear heat conduction equation.

I. We consider a thermal process described by the equation

$$C(T)\frac{\partial T}{\partial t} = \nabla(\lambda(T)\nabla T) + Q(T),\ (x,\ t)\in\Omega \tag{1}$$

with initial and boundary conditions

$$T(x,\ 0) = 0,\ x\in D, \tag{2}$$

$$\begin{gathered} T(0,\ x_2,\ x_3,\ \ldots,\ x_n,\ t) = f_1 t^m, \\ T(x_1,\ 0,\ x_3,\ \ldots,\ x_n,\ t) = f_2 t^m, \\ \cdots\cdots\cdots\cdots \\ T(x_1,\ x_2,\ x_3,\ \ldots,\ x_{n-1},\ 0,\ t) = f_n t^m,\ t>0, \end{gathered} \tag{3}$$

where f_i, $i = \overline{1,\ n}$, and m are given constants; $n \geq 2$ is a given positive integer. With no loss of generality we assume that $f_i > 0$.

Before presenting a statement of the inverse problem we indicate sufficient conditions for guaranteeing self-similarity for a solution of boundary value problem (1)-(3). We assume the following condition A to be satisfied: if m = 0, then Q(T) = 0; but if $m \neq 0$, then C(T) and λ(T) are homogeneous functions of order $\sigma \neq 0$ and Q(T) is a homogeneous function of order $\sigma + 1 - 1/m$.

We seek a solution of Eq. (1), subject to conditions (2) and (3), in the form $T(x,\ t) = t^m v(z)$, where $z = (z_1,\ z_2,\ \ldots,\ z_n)$, $z_i = x_i t^{-1/2}$, $i = \overline{1,\ n}$. We then find that under assumption A the function v(z) satisfies the equation

$$mvC(v) - 0{,}5C(v)\sum_{i=1}^{n} z_i\frac{\partial v}{\partial z_i} = \sum_{i=1}^{n}\frac{\partial}{\partial z_i}\left[\lambda(v)\frac{\partial v}{\partial z_i}\right] + Q(v),\ z\in D \tag{4}$$

and the conditions

Bakinskii State University. Translated from Inzhenerno-fizicheskii Zhurnal, Vol. 61, No. 3, pp. 472-478, September, 1991. Original article submitted November 27, 1990.

$$v(\infty, \infty, \ldots, \infty) = 0, \tag{5}$$

$$\begin{gathered} v(0, z_2, z_3, \ldots, z_{n-1}, z_n) = f_1, \\ v(z_1, 0, z_3, \ldots, z_{n-1}, z_n) = f_2, \\ \ldots\ldots\ldots\ldots\ldots \\ v(z_1, z_2, z_3, \ldots, z_{n-1}, 0) = f_n. \end{gathered} \tag{6}$$

Our interest centers on a solution of inverse problems concerned with determining the coefficients of Eq. (1). It is obvious that if the coefficients of Eq. (1) are unknown, the conditions (1)-(3) are then insufficient for their determination. It is therefore necessary to append additional conditions to the system (1)-(3). We give several such typical conditions usually used in solving inverse problems:

$$T(x, t_N) = \psi_0(x), \ x \in D, \tag{7}$$

$$T(\eta, x', t) = \psi_1(x', t), \ x' \in D', \ t > 0, \tag{8}$$

$$\int_0^l T(x_1, x', t)\, dx_1 = \psi_2(x', t), \ x' \in D', \ t > 0. \tag{9}$$

Each of the conditions (7)-(9) has a real physical meaning. Function $\psi_0(x)$ gives the temperature distribution in the body at time $t = t_N$; function $\psi_1(x', t)$ is the value of the temperature on the curve $x_1 = \eta$ for arbitrary time $t > 0$; and function $\psi_2(x', t)$ is connected with the expression for the general quantity of heat contained in domain $(0, l)$ for arbitrary time $t > 0$. It is readily seen that for each of the functions $\psi_0(x)$, $\psi_1(x', t)$, $\psi_2(x', t)$ a self-similar solution of Eq. (1) (see [7]) can be determined uniquely:

$$\begin{gathered} v(z) = t_N^{-m}\psi_0(\sqrt{t_N}z_1, \sqrt{t_N}z_2, \ldots, \sqrt{t_N}z_n) = \\ = \left(\frac{z_1}{\eta}\right)^{2m}\psi_1\left(\frac{\eta z_2}{z_1}, \frac{\eta z_3}{z_1}, \ldots, \frac{\eta z_n}{z_1}, \frac{\eta^2}{z_1^2}\right) = \\ = \frac{2m+1}{l}\left(\frac{l}{z_1}\right)^{1-2m}\psi_2\left(\frac{lz_2}{z_1}, \frac{lz_3}{z_1}, \ldots, \frac{lz_n}{z_1}, \frac{l^2}{z_1^2}\right) + \\ + \frac{4l}{z_1^3}\left(\frac{l}{z_1}\right)^{2-2m}\frac{\partial\psi_2}{\partial z_1}\left(\frac{lz_2}{z_1}, \frac{lz_3}{z_1}, \ldots, \frac{lz_n}{z_1}, \frac{l^2}{z_1^2}\right), \\ z_i = \frac{x_i}{\sqrt{t}} > 0. \end{gathered} \tag{10}$$

Conditions different from relations (7)-(9) also exist, which make it possible to determine a self-similar solution of Eq. (1). To avoid considering each of these cases in detail, we assume that a self-similar solution of Eq. (1) is given and that one or several coefficients of Eq. (1) are to be determined.

Let $C(T) > 0$ $Q(T) \le 0$ be specified on $(0, +\infty)$ and let these functions be bounded and continuous. From conditions (1)-(3), with the twice continuously differentiable function $y(z_1) = v(z_1, z_2^{(0)}, z_3^{(0)}, \ldots, z_n^{(0)})$ given, we are required to determine coefficient $\lambda(T)$, continuous, positive, and bounded on $(0, \infty)$, where $z_2^{(0)}, z_3^{(0)}, \ldots, z_n^{(0)}$ are given positive numbers. We assume, in addition, that $\lambda(0) = \kappa_0 > 0$, $v_{z_i}(z_1, \xi_0)$, $v_{z_i z_i}(z_1, \xi_0)$, $i = \overline{2, n}$, are also given, where $\xi_0 = (z_2^{(0)}, z_3^{(0)}, \ldots, z_n^{(0)})$. We introduce the notation: $\alpha_i(z_1) = v_{z_i}(z_1, \xi_0)$, $\beta_i(z_1) = v_{z_i z_i}(z_1, \xi_0)$, $i = \overline{2, n}$, $f_0 = \max\{f_1, f_2, \ldots, f_n\}$.

Let the following conditions by satisfied:

1) $y(z_1)$ has an inverse function $F(y)$, defined on $(0, f_0]$ with domain values on $[0, +\infty)$;

2) $$F_y^{-2}(y) + \sum_{i=0}^{n}\alpha_i^2(F(y)) \neq 0, \ y \in (0, f_0].$$

We then have the following expression for function $\lambda(T)$:

$$\lambda(T) = \exp\left[-\int_0^T P(s)\,ds\right]\left[\varkappa_0 + \int_0^T R(s)\exp\left[\int_0^s P(\zeta)\,d\zeta\right]ds\right], \tag{11}$$

where

$$P(y) = \left[\sum_{i=2}^{n} \beta_i(F(y)) - \frac{F_{yy}(y)}{F_y^3(y)}\right]\left[\frac{1}{F_y^2(y)} + \sum_{i=2}^{n} \alpha_i(F(y))\right]^{-1},$$

$$R(y) = \left[myC(y) - Q(y) - \frac{0{,}5C(y)F(y)}{F_y(y)} - 0{,}5C(y)\sum_{i=2}^{n} z_i^{(0)}\alpha_i(F(y))\right] \times$$

$$\times \left[\frac{1}{F_y^2(y)} + \sum_{i=2}^{n} \alpha_i(F(y))\right]^{-1}.$$

Actually, Eq. (4) can be written in the form

$$\sum_{i=1}^{n}\left(\frac{\partial v}{\partial z_i}\right)^2 \lambda'(v) + \sum_{i=1}^{n} \frac{\partial^2 v}{\partial z_i^2}\lambda(v) + Q(v) =$$

$$= mvC(v) - 0{,}5C(v)\sum_{i=1}^{n} z_i \frac{\partial v}{\partial z_i}, \quad z \in D. \tag{12}$$

This equation is valid for arbitrary $z \in D$. We write Eq. (12) at the point $(z_1, \xi_0) \in D$; in the resulting equation we pass over to the inverse function $z_1 = F(y)$ and we consider that

$$v_{z_1}(z_1, \xi_0) = F_y^{-1}(y), \quad v_{z_1 z_1}(z_1, \xi_0) = -F_y^{-3}(y)F_{yy}(y).$$

Then in the notation adopted above Eq. (12) is transformed to the form

$$\left[\frac{1}{F_y^2(y)} + \sum_{i=2}^{n} \alpha_i(F(y))\right]\lambda'(y) + \left[\sum_{i=2}^{n} \beta_i(F(y)) - \frac{F_{yy}(y)}{F_y^3(y)}\right]\lambda(y) =$$

$$= myC(y) - Q(y) - 0{,}5C(y)\left[\frac{F(y)}{F_y(y)} + \sum_{i=1}^{n} z_i^{(0)}\alpha_i(F(y))\right].$$

From this we have

$$\lambda'(y) + P(y)\lambda(y) = R(y). \tag{13}$$

A solution of this ordinary differential equation, with the condition $\lambda(0) = \kappa_0$, has the form (11). This establishes the validity of formula (11).

But if coefficient $\lambda(T)$ is given and it is coefficient $C(T)$ that is being sought, then under the assumptions enumerated above we have the expression

$$C(T) = \left\{\left[\frac{1}{F_T^2(T)} + \sum_{i=2}^{n} \alpha_i^2(F(T))\right]\lambda'(T) + \left[\sum_{i=2}^{n} \beta_i(F(T)) - \right.\right.$$

$$\left.\left. - \frac{F_{TT}(T)}{F_T^3(T)}\right]\lambda(T) + Q(T)\right\}\left[mT - \frac{F(T)}{2F_T(T)} - \frac{1}{2}\sum_{i=2}^{n} z_i^{(0)}\alpha_i(F(T))\right]^{-1}. \tag{14}$$

The right side of Eq. (14) is assumed to be continuous and positive. A similar expression holds also for coefficient $Q(T)$ if $\lambda(T)$ and $C(T)$ are given.

II. Assume now that we wish to determine coefficients $C(T) > 0$, $\lambda(T) > 0$, $Q(T)$ of Eq. (1) simultaneously. We consider the cases $m = 0$ and $m \neq 0$ separately.

1°. Assume $m = 0$. According to condition A, in this case we must put $Q(T) = 0$. It is obvious that to different initial data f_i, $i = \overline{1, n}$ in condition (6) there correspond different solutions of Eq. (4). We assume that the numbers f_{1i}, f_{2i}, $i = \overline{1, n}$ correspond to self-similar solutions $v_1(z)$, $v_2(z)$ of Eq. (4). Let the functions $y = v_1(z_1, \xi_1)$, $y = v_2(z_1, \xi_2)$ be given and suppose we wish to determine the unknown coefficients $\lambda(T) > 0$ and $C(T) > 0$, where $\xi_1 = (z_2^{(1)}, z_3^{(1)}, \ldots, z_n^{(1)})$, $\xi_2 = (z_2^{(2)}, z_3^{(2)}, \ldots, z_n^{(2)})$; in particular, we may have $\xi_1 \equiv \xi_2$. In addition, we assume that $v_{kz_i}(z_1, \xi_k)$, $v_{kz_iz_i}(z_1, \xi_k)$, $k = 1, 2$, $i = \overline{2, n}$, $\lambda(0) = \kappa_0 > 0$ are also given. Let $F_k(y)$ be the function inverse to the function $v_k(z_1, \xi_k)$,

$$M(y) = \left\{ \sum_{k=1}^{2} (-1)^{k-1} \left[\sum_{i=2}^{n} \beta_{ki} (F_k(y)) - \frac{F_{kyy}(y)}{F_{ky}^3(y)} \right] \left[\sum_{i=2}^{n} z_i^{(k)} \alpha_{ki} (F_k(y)) + \frac{F_k(y)}{F_{ky}(y)} \right] \right\} \left\{ \sum_{k=1}^{2} (-1)^{k-1} \left[\frac{1}{F_{ky}^2(y)} + \sum_{i=2}^{n} \alpha_{ki} (F_k(y)) \right] \left[\sum_{i=2}^{n} z_i^{(k)} \alpha_{ki} (F_k(y)) + \frac{F_k(y)}{F_{ky}(y)} \right]^{-1} \right\}^{-1}, \quad N(y) = \left\{ - \left[\frac{1}{F_{1y}^2(y)} + \sum_{i=2}^{n} \alpha_{1i} (F_1(y)) \right] M(y) - \left[\sum_{i=2}^{n} \beta_{1i} (F_1(y)) - \frac{F_{1yy}(y)}{F_{1y}^3(y)} \right] \right\} \left[\sum_{i=2}^{n} z_i^{(1)} \alpha_{1i} (F_1(y)) + \frac{F_1(y)}{F_{1y}(y)} \right]^{-1},$$

$$\alpha_{ki}(z_1) = v_{kz_i}(z_1, \xi_k), \ \beta_{ki}(z_1) = v_{kz_iz_i}(z_1, \xi_k), \ i = \overline{2, n},$$

$$f_k = \max\{f_{k1}, f_{k2}, \ldots, f_{kn}\}, \ k = 1, 2.$$

Assume $m = 0$ and that the following conditions are satisfied:

1) Function $y = v_k(z_1, \xi_k)$ has an inverse function $F_k(y)$, defined on $(0, f_k]$ with domain of values on $(0, \infty)$;

2) $$F_k(y) \neq 0, \ F_{ky}(y) \neq 0, \ \sum_{i=2}^{n} z_i^{(k)} \alpha_{ki} (F_k(y)) + \frac{F_k(y)}{F_{ky}(y)} \neq 0, \ y \in (0, f_k], \ k = 1, 2;$$

3) $$\sum_{k=1}^{2} (-1)^{k-1} \left[\frac{1}{F_{ky}^2(y)} + \sum_{i=2}^{n} \alpha_{ki} (F_k(y)) \right] \left[\sum_{i=2}^{n} z_i^{(k)} \alpha_{ki} \times (F_k(y)) + \frac{F_k(y)}{F_{ky}(y)} \right]^{-1} \neq 0.$$

We then have the following expressions for functions $\lambda(T)$ and $C(T)$:

$$\lambda(T) = \varkappa_0 \exp\left[- \int_0^T M(s)\, ds \right], \ T \in (0, f_1], \tag{15}$$

$$C(T) = -2\varkappa_0 N(T) \exp\left[\int_0^T M(S)\, dS \right], \ T \in (0, f_1]. \tag{16}$$

Actually, we rewrite Eq. (4) for $k = 1, 2$ in the form

$$\sum_{i=1}^{n} \left(\frac{\partial v_k}{\partial z_i} \right)^2 \lambda'(v_k) + \sum_{i=1}^{n} \frac{\partial^2 v_k}{\partial z_i^2} \lambda(v_k) + 0{,}5 \sum_{i=1}^{n} z_i \frac{\partial v_k}{\partial z_i} C(v_k) = 0, \ k = 1{,}2. \tag{17}$$

This is a system of ordinary differential equations with respect to $\lambda(y)$ with unknown function $C(y)$. In Eq. (17) we go over to the inverse function $z_1 = F_k(y)$ and consider that $v_{kz_1}(z_1, \xi_k) = F_{ky}^{-1}(y)$, $v_{kz_1z_1}(z_1, \xi_k) = -F_{kyy}(y) F_{ky}^{-3}(y)$. Then in the notation adopted above we have

$$\left[\frac{1}{F_{ky}^2(y)} + \sum_{i=2}^{n} \alpha_{ki} (F_k(y)) \right] \left[\sum_{i=2}^{n} z_i^{(k)} \alpha_{ki} (F_k(y)) + \frac{F_k(y)}{F_{ky}(y)} \right]^{-1} \lambda'(y) +$$

$$+\left[\sum_{i=2}^{n}\beta_{hi}(F_h(y))-\frac{F_{hyy}(y)}{F_{hy}^3(y)}\right]\left[\sum_{i=2}^{n}z_i^{(k)}\alpha_{hi}(F_h(y))+\frac{F_h(y)}{F_{hy}(y)}\right]^{-1}\lambda(y)+$$

$$+0{,}5C(y)=0,\ k=1{,}2. \tag{18}$$

From the equation for k = 1 we subtract the equation for k = 2. Then using the notation adopted above, we obtain

$$\lambda'(y)+M(y)\lambda(y)=0.$$

A solution of this ordinary differential equation, with the condition $\lambda(0) = \kappa_0$ has the form (15). Substituting the expression obtained for $\lambda(y)$ into Eqs. (18), we deduce that expression (16) is valid. Thus we have shown that formulas (15) and (16) are valid.

2°. Now let $m \neq 0$. According to condition A, in this case, $C(T) = c_0T^{\sigma}$, $\lambda(T) = \lambda_0T^{\sigma}$, $Q(T) = Q_0T^{\sigma+1-1/m}$, where $c_0 > 0$, $\lambda_0 > 0$, $Q_0 \le 0$, and σ are certain numbers. Assume we wish to determine coefficients $C(T) > 0$, $\lambda(T) > 0$, $Q(T) \le 0$ of Eq. (1); this leads to obtaining unknown constants in the expressions for these coefficients. We assume that $c_0 > 0$ is a given constant and we wish to determine $\lambda_0 > 0$, σ, $Q_0 \le 0$. Let

$$z^{(k)}=(z_1^{(k)},\ z_2^{(k)},\ \ldots,\ z_n^{(k)}),\ \alpha_{k1}=\sum_{i=1}^{n}[v_{z_i}(z^{(k)})]^2,$$

$$\alpha_{k2}=v(z^{(k)})\sum_{i=1}^{n}v_{z_iz_i}(z^{(k)}),\ \alpha_{k3}=[v(z^{(k)})]^{2-1/m},$$

$$\beta_k=c_0m[v(z^{(k)})]^2-0{,}5c_0v(z^{(k)})\sum_{i=1}^{n}z_i^{(k)}v_{z_i}(z^{(k)}),\ k=1,\ 2,\ 3,$$

$$\alpha=(\alpha_{12}\alpha_{33}-\alpha_{13}\alpha_{32})(\alpha_{21}\alpha_{13}-\alpha_{11}\alpha_{23})-$$
$$-(\alpha_{12}\alpha_{23}-\alpha_{22}\alpha_{13})(\alpha_{31}\alpha_{13}-\alpha_{11}\alpha_{33}),$$
$$\beta=(\beta_2\alpha_{13}-\beta_1\alpha_{23})(\alpha_{31}\alpha_{13}-\alpha_{12}\alpha_{33})-$$
$$-(\beta_3\alpha_{13}-\beta_1\alpha_{33})(\alpha_{22}\alpha_{13}-\alpha_{11}\alpha_{33}).$$

Assume that at the three arbitrary points $z^{(1)}$, $z^{(2)}$, $z^{(3)}$ we are given values of $v(z)$, $v_{z_i}(z)$, $v_{z_iz_i}(z)$, $i = \overline{1, n}$, and $\alpha\beta > 0$, $\alpha_{21}\alpha_{13} \neq \alpha_{11}\alpha_{23}$, $\det\|\alpha_{ks}\| \neq 0$, $k, s = 1, 3$, β_k, $k = 1, 3$, do not vanish simultaneously.

Equation (4) is true for arbitrary $z \in D$ and $C(v) > 0$, $\lambda(v) > 0$, $Q(v) \le 0$. If we write down Eq. (4) for $C(v) = c_0v^{\sigma}$, $\lambda(v) = \lambda_0v^{\sigma}$, $Q(v) = Q_0v^{\sigma+1-1/m}$ at the points $z^{(1)}$, $z^{(2)}$, $z^{(3)}$ and use the notation adopted above, we then find that

$$\lambda_0\sigma\alpha_{k1}+\lambda_0\alpha_{k2}+Q_0\alpha_{k3}=\beta_k,\ k=1,\ 2,\ 3.$$

Under the assumptions adopted above, this system has a unique solution:

$$\lambda_0=\beta/\alpha,\ \sigma=[\beta(\alpha_{12}\alpha_{23}-\alpha_{22}\alpha_{13})+\alpha(\beta_2\alpha_{13}-\alpha_{23}\beta_1)]/\beta(\alpha_{21}\alpha_{13}-$$
$$-\alpha_{11}\alpha_{23}),\ Q_0=[\alpha(\beta_1\alpha_{21}-\alpha_{11}\beta_2)+\beta(\alpha_{11}\alpha_{22}-$$
$$-\alpha_{12}\alpha_{21})]/\alpha(\alpha_{21}\alpha_{13}-\alpha_{11}\alpha_{23}).$$

TABLE 1. Comparison of Exact and Approximate Values of Thermal Conductivity Coefficient

λ	T									
	0,1653	0,2019	0,2466	0,3012	2,3679	0,4493	0,5488	0,6703	0,8187	1,000
I	0,70	0,65	0,60	0,55	0,50	0,45	0,40	0,35	0,30	0,25
II	0,7195	0,6681	0,6172	0,5624	0,5095	0,4610	0,4096	0,3520	0,3520	0,2489

An analogous formula holds if, instead of λ_0, σ, Q_0, we seek other numerical parameters.

The explicit expressions obtained in sections I and II make it possible to work out simple stable algorithms for an approximate solution of the inverse problems considered above [10].

Example. Consider the determination of coefficient $\lambda(T) > 0$ from conditions (10-(3) when $n = 2$, $C(T) = 1$, $Q(T) = 0$, $m = 0$, $t_N = 1$, $T(x_1, x_2, t_N) = \psi_0(x_1, x_2) = \exp(-x_1, -x_2)$. It follows from Eq. (11) that $\lambda(T) = (1 - \ln T)/4$, $0 \leq T \leq 1$. A direct verification shows that in the given case $T(x_1, x_2, t) = [(-x_1 - x_2)/\sqrt{t}]$, $v(z_1, z_2) = \exp(-z_1 - z_2)$, $\lambda(T) = (1 - \ln T)/4$ satisfy the conditions for systems (1)-(3) and (4)-(6).

In Table 1 we have shown for comparison the exact and approximate values of coefficient $\lambda(T)$ at the nodes of a nonuniform grid introduced in the interval [0, 1]. Here use has been made of relation (12). The first and second rows of Table 1 give the exact and the approximate values, respectively, for $\lambda(T)$; the approximate values are obtained from relation (12).

NOTATION

t, time; t_N, time interval of observation; D, domain in n-dimensional euclidean space; $D \equiv \{x : x \equiv (x_1, x_2, \ldots, x_n), x_i > 0, i = \overline{1, n}\}$; D', domain in (n − 1)-dimensional euclidean space; $D' = \{x' : x' = (x_2, x_3, \ldots, x_n), x_i > 0, i = \overline{2, n}\}$; $x = (x_1, x_2, \ldots, x_n)$, arbitrary point of domain D; $x' = (x_2, x_3, \ldots, x_n)$, arbitrary point of domain D'; T, temperature field; C, volume heat capacity; λ, thermal conductivity coefficient; Q, strength of internal sources; η, ℓ, fixed positive points; $\Omega \equiv D \times (0, t_N]$, $\nabla \equiv [(\partial/\partial x_1), (\partial/\partial x_2), \ldots, (\partial/\partial x_n)]$.

LITERATURE CITED

1. A. D. Iskenderov, Dokl. Akad. Nauk SSSR, 225, No. 5, 1005-1008 (1975).
2. O. M. Alifanov, Inverse Problems of Heat Transfer [in Russian], Moscow (1988).
3. O. M. Alifanov, E. A. Artyukhin, and S. V. Rumyantsev, Extremal Methods for Solving Ill-Posed Problems [in Russian], Moscow (1988).
4. M. V. Klibanov, Dokl. Akad. Nauk SSSR, 280, No. 3, 533-536 (1985).
5. A. D. Iskenderov and A. A. Akhundov, Izv. Akad. Nauk Az. SSR, Ser. Fiz.-Tekh. Mat. Nauk, No. 3, 82-85 (1976).
6. A. D. Iskenderov and T. B. Gardashov, Dokl. Akad. Nauk Az, SSR, 43, No. 2, 17-20 (1987).
7. A. D. Iskenderov and T. B. Gardashov, Differents. Uravn., 26, No. 7, 1148-1153 (1990).
8. A. D. Iskenderov, Dzh. F. Dzhafarov, and T. B. Gardashov, Inzh.-fiz. Zh., 46, No. 6, 1030-1031 (1984).
9. A. D. Iskenderov, T. B. Gardashov, and T. M. Ibragimov, Inzh.-fiz. Zh., 56, No. 1, 127-132 (1989).
10. A. N. Tikhonov and V. Ya. Arsenin, Methods for Solving Ill-Posed Problems [in Russian], Moscow (1986).